Year 2023-2024

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis

Homework 3: Martingales

Due Date: 23:59 pm on Tuesday, February 6th, 2024.
Please submit your homework on Blackboard

1. (a) Let $X : \Omega \to \mathbb{R}$ be a random variable such that $X \equiv 0$, i.e. for any $\omega \in \Omega$, $X(\omega) = 0$. Prove that $\sigma(X) = \{\emptyset, \Omega\}$.

   (b) Let $\mathcal{G} := \{\emptyset, \Omega\}$, and $X : \Omega \to \mathbb{R}$ be $\mathcal{G}$-measurable. Prove that $X \equiv c$ for some constant $c \in \mathbb{R}$.

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ be a filtration. Given an $\mathbb{F}$-predictable process $(H_n)_{n\geq 0}$, which is uniformly bounded, and an $\mathbb{F}$-martingale $(X_n)_{n\geq 0}$, we define a process $(V_n)_{n\geq 0}$ by

$$V_0 := 0, \quad V_n := \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

   Prove that $(V_n)_{n\geq 0}$ is still an $\mathbb{F}$-martingale.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ be a filtration. Given an $\mathbb{F}$-submartingale $(X_n)_{n\geq 0}$, we define

$$\Delta A_n := \mathbb{E}[X_n \mid \mathcal{F}_{n-1}] - X_{n-1}, \quad \Delta M_n := X_n - \mathbb{E}[X_n|\mathcal{F}_{n-1}], \quad \forall n \geq 1,$$

   and

$$A_0 = M_0 = 0, \quad A_n := \sum_{k=1}^{n} \Delta A_k, \quad M_n := \sum_{k=1}^{n} \Delta M_k.$$

   (a) Prove that $(M_n)_{n\geq 0}$ is an $\mathbb{F}$-martingale, and that $(A_n)_{n\geq 0}$ is an increasing $\mathbb{F}$-predictable process.

   (b) Prove that $(X_n)_{n\geq 0}$ has the decomposition

$$X_n = X_0 + M_n + A_n, \quad \forall n \geq 0. \tag{1}$$

(c) Let $(A^1_n)_{n\geq 0}$ and $(A^2_n)_{n\geq 0}$ be two $\mathbb{F}$-predictable processes such that $A^1_0 = A^2_0 = 0$. Prove that if $(A^1_n - A^2_n)_{n\geq 0}$ is an $\mathbb{F}$-martingale, then $A^1_n = A^2_n$, a.s. for each $n \geq 1$.

(d) Deduce that the decomposition (1) is unique, i.e. if one has

$$X_n = X_0 + \widetilde{M}_n + \widetilde{A}_n, \quad \forall n \geq 0.$$

for some $\mathbb{F}$-martingale $(\widetilde{M}_n)_{n\geq 0}$ and increasing $\mathbb{F}$-predictable process $(\widetilde{A}_n)_{n\geq 0}$ such that $\widetilde{M}_0 = \widetilde{A}_0 = 0$, then $A_n = \widetilde{A}_n$ and $M_n = \widetilde{M}_n$, a.s. for each $n \geq 1$.